Midterm Supplementary - Optimization (2019-20)

Attempt all questions. There are a total of 27 points, the maximum you can score is 25.

Time: 2 hours.

1. Show $< \mathbf{x}, \mathbf{y} >= \mathbf{y}^{\mathbf{T}}\mathbf{A}\mathbf{x}$ on $\mathbb{R}^3$ is an inner product, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}. \qquad [\textbf{4 points}]$$

2. Consider the problem: minimize $2|x_1| + x_2$ subject to $x_1 + x_2 \geq 4$. Reformulate this as a linear programming problem. [**3 points**]

3. For each of the statements below, state whether it is true or false. If true, prove it and if false, give a counterexample.

   (a) An unbounded polyhedron cannot have more than one extreme point. [**3 points**]

   (b) Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Assume $\mathbf{A}$ has dimensions $m \times n$ and the rows of $\mathbf{A}$ are linearly independent. For every optimal solution, no more than m variables can be positive. [**3 points**]

   (c) Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron P. The set of optimal solutions is bounded. [**3 points**]

   (d) Consider the problem of minimizing $\mathbf{c}^T\mathbf{x}$ over a polyhedron $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Assume $\mathbf{A}$ has dimensions $m \times n$ and the rows of $\mathbf{A}$ are linearly independent. A degenerate basic solution corresponds to two or more distinct bases. [**3 points**]

4. Suppose that the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i^T\mathbf{x} \geq b_i,\ i = 1, 2, \cdots, m\}$ is nonempty. If there exist n vectors out of the family $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_m$ which are linearly independent, then show that P does not contain a line. [**4 points**]

5. Let P be a bounded polyhedron in $\mathbb{R}^n$, let $\mathbf{a}$ be a vector in $\mathbb{R}^n$, and let b be some scalar. We define

$$Q = \{\mathbf{x} \in P : \mathbf{a}^T\mathbf{x} = b\}.$$

   Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P. [**4 points**]
   Note: $\mathbf{x}$ *and* $\mathbf{y}$ *are adjacent extreme points if there are* $n-1$ *linearly independent constraints active at both* $\mathbf{x}$ *and* $\mathbf{y}$.